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Extensions of Exchange Rings*

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We define non-unital exchange rings and we prove that if I is an ideal of a ring R, then R is an exchange ring if and only if I and R/I are exchange rings and idempotents can be lifted modulo I. We also show that we can replace the condition on liftability of idempotents with the condition that the canonical map $K_0(R) \to K_0(R/I)$ be surjective. © 1997 Academic Press

INTRODUCTION

Given an ideal I of a ring R, it is generally useful to be able to determine properties of R from knowledge of properties of I and R/I. This kind of question is generically known as an *extension problem*. For instance, McCoy's Lemma which states that a ring R is von Neumann regular if and only if I and R/I are, is widely used in specific computations with regular rings; see, for example, [8]. Extension problems are also common in the field of operator algebras; see, for example, [4, Chap. VII]. In [5, Theorem 3.14], Brown and Pedersen showed that if I is a closed ideal of a C^*-algebra A, then A has real rank zero if and only if the real ranks of I and A/I are zero and every projection (self-adjoint idempotent) in A/I lifts to a projection in A.

An associative unital ring R is said to be an *exchange ring* [19] if R_R has the exchange property introduced by Crawley and Jónsson [7]. Warfield proved that this property is left-right symmetric. The structure of exchange rings has been investigated by numerous authors, most recently in [21, 22, 6, 1, 2, 15, 3].

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The class of exchange rings is quite large. It includes all semiperfect rings, all von Neumann regular rings, and all π-regular rings; see [19, 18]. Moreover, by using (among other things) the main result of the present paper, Baccella has recently proved that all right or left semiartinian rings are exchange rings [3]. An interesting connection with operator algebras has been discovered in [2], where it is proved that the unital C^*-algebras that are exchange rings are precisely the C^*-algebras of real rank zero. This result raises the question of determining what structural properties of C^*-algebras of real rank zero generalize to exchange rings, and suggests the possibility of using techniques from the theory of exchange rings in the study of C^*-algebras of real rank zero.

Let R be a ring and I an ideal of R. This paper gives a complete answer to the following problem: What properties of I and R/I determine whether R is an exchange ring? It is natural to expect that the solution involves both I and R/I being exchange rings. However, this concept has not been previously studied in the non-unital case, so, in Section 1, we adapt a characterization from the unital case, and develop a few basic properties needed for the proof of our main theorem. In detail, a ring without unit I is said to be an *exchange ring* if for each $x \in I$ there exist an idempotent $e \in I$ and $r, s \in I$ such that $e = xr = x + s - xs$. As in the unital case, this property is left-right symmetric, and it gives exchange properties for suitable module decompositions (see Theorem 1.2). Every (two-sided) ideal of an exchange ring with unit is an exchange ring, but there exist exchange rings without unit which are not ideals of any unital exchange ring.

An extension of a ring S by a ring I is a ring R such that I is an ideal of R, and $R/I \cong S$. (We should remark that this terminology is the common usage in the theory of operator algebras, but the reverse terminology is used in group theory and module theory.) We prove in Section 2 that such an extension is an exchange ring if and only if S and I are exchange rings and idempotents can be lifted from R/I to R. This was proved by Nicholson in the special case where R is unital and I is contained in the Jacobson radical of R [14, Proposition 1.5] (in this case I is automatically an exchange ring). Since idempotents can always be lifted modulo a π-regular ideal [12, Lemma 3], we obtain as a consequence that every extension of an exchange ring by a π-regular ring is again an exchange ring.

Finally, in Section 3 we prove that lifting of idempotents modulo an exchange ideal can be checked from K-theoretic data. As an application, we show that an extension of a purely infinite right self-injective ring by an exchange ring always inherits the exchange property (Corollary 3.6). We also obtain the extension theorems of Brown and Pedersen on C^*-algebras of real rank zero [5, 3.14 and 3.15] as corollaries of our results.

In order to avoid confusion, from now on we will denote rings without unit by I or L, while R, S, T will always denote unital rings. All our modules over unital rings will be unital (i.e., $m = m \cdot 1_R$ for all $m \in M_R$). If I is a ring, we denote by I^1 the unitization of I; that is, $I^1 = I \oplus \mathbb{Z}$ with addition and multiplication defined by $(x, n) + (y, m) = (x + y, n + m)$, and $(x, n)(y, m) = (xy + ny + mx, nm)$, for all $x, y \in I$ and $n, m \in \mathbb{Z}$.

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1. EXCHANGE RINGS WITHOUT UNIT

In this section, we introduce the notion of exchange rings without unit. It turns out that most of the properties of unital exchange rings hold, with suitable modifications, in this broader context.

The proof of the following lemma is similar to the one in [14, Proposition 1.1].

LEMMA 1.1. *Let I be a ring without unit and let R be a unital ring containing I as a (two-sided) ideal. Then the following conditions are equivalent for an element $x \in I$:*

(1) *There exists $e^2 = e \in I$ with $e - x \in R(x - x^2)$.*

(2) *There exists $e^2 = e \in Ix$ and $c \in R$ such that $(1 - e) - c(1 - x) \in J(R)$.*

(3) *There exists $e^2 = e \in Ix$ such that $R = Ie + R(1 - x)$.*

(4) *There exists $e^2 = e \in Ix$ such that $1 - e \in R(1 - x)$.*

(5) *There exist $r, s \in I$ and $e = e^2 \in I$ such that $e = rx = s + x - sx$.*

Let I be a ring without unit and let R be a unital ring containing I as an ideal. Let M_S be a module over a unital ring S such that there exists an isomorphism $\varphi\colon \operatorname{End}(M_S) \to R$. Let

$$X = M \oplus Y = N_1 \oplus N_2 \qquad (*)$$

be two decompositions of a right S-module X. Denote by π the idempotent in $\operatorname{End}(X)$ with image M and kernel Y, and identify $\operatorname{End}(M)$ with $\pi \operatorname{End}(X)\pi$. We say that $(*)$ is *I-admissible* if $\pi\tau_2\pi \in \varphi^{-1}(I)$, where $\tau_i \in \operatorname{End}(X)$ is the projection onto N_i determined by the decomposition $X = N_1 \oplus N_2$.

The proof of the following theorem is similar to that of [14, Theorem 2.1].

THEOREM 1.2. *Let* (I, R) *and* $\varphi\colon \mathrm{End}(M_S) \to R$ *be as above. Then the following conditions are equivalent*:

(a) *For all* $x \in I$, *there exist* $e = e^2 \in I$ *and* $r, s \in I$ *such that* $e = rx = x + s - sx$.

(b) *For all I-admissible decompositions*

$$X = M \oplus Y = N_1 \oplus N_2 \qquad (*)$$

there exist $N_i' \subseteq N_i$ *such that* $X = M \oplus N_1' \oplus N_2'$.

(c) *For all* $x \in I$, *there exist* $e = e^2 \in I$ *and* $r, s \in I$ *such that* $e = xr = s + x - xs$.

DEFINITION. A ring without unit I is an *exchange ring* if it satisfies the equivalent conditions in Theorem 1.2. Note that I being an exchange ring is a symmetric condition which does not depend on the particular unital ring where I is embedded as an ideal. (Look at conditions (a) and (c) in Theorem 1.2.)

EXAMPLES.

(1) If I is an ideal of a unital exchange ring, then I is an exchange ring. To see this, take an element $x \in I$. By [14, Theorem 2.1], there exist $r, s \in R$ and an idempotent $e \in R$ such that $e = xr$ and $1 - e = (1 - x)(1 - s)$. Clearly $e \in I$, and therefore $s = e - x + xs \in I$. Consequently we can write $e = x(re) = x + s - xs$, with $re, s \in I$ and condition (c) of Theorem 1.2 is satisfied.

(2) The *radical rings* [10] are exactly the exchange rings without nonzero idempotents.

(3) A ring I is said to be π-regular if for all $x \in I$ there exist a positive integer n and $y \in I$ such that $x^n = x^n y x^n$. All the (non-unital) π-regular rings are exchange rings, by the proof of [18, Example 2.3].

(4) There exist exchange rings without unit which are not ideals of unital exchange rings. For an example, take $I = \left(\begin{smallmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right)$. Then I is an exchange ring because $I^2 = 0$. Assume that R is a unital ring that contains I as an ideal. Then there exists a unital ring homomorphism $\varphi\colon R \to \mathrm{End}_{\mathbb{Z}}(I)$. Now $\mathrm{End}_{\mathbb{Z}}(I) \cong \mathbb{Z}$, and so φ is surjective. Therefore $\mathbb{Z}$ is a homomorphic image of R, and so R is not an exchange ring [14, Proposition 1.4].

The following easy proposition is very useful. It is the analogue of [14, Proposition 1.10]. Usually we will consider a non-unital exchange ring I which is an ideal of the ring R. We will shorten this situation saying that "I is an exchange ideal of R." However, we remark again that being an exchange ring is an intrinsic property of the ring without unit I.

PROPOSITION 1.3. *Let I be an exchange ideal of the ring R, and let e be an idempotent of R. Then eIe is an exchange ideal of eRe.*

Proof. Let $x \in eIe$. Then $x \in I$ and so there exists an idempotent $f \in Ix$ such that $1 - f \in R(1 - x)$. Note that $fe = f$. So ef is an idempotent in eIe and $ef \in (eIe)x$. Now write $1 - f = s(1 - x)$ for some $s \in R$. We obtain $e - ef = e - efe = es(1 - x)e = (ese)(e - x) \in (eRe)(e - x)$. So eIe is an exchange ideal of eRe. ∎

Since the following result is crucial for our work, we will include the proof, which is an amalgamation of the proofs of [14, Theorem 2.1; 7, Lemma 3.10].

THEOREM 1.4. *Let I be a ring, possibly without unit. If I is an exchange ring, then $M_n(I)$ is an exchange ring for all $n \geq 1$.*

Remark. The result is well known for unital exchange rings. See [14, 2.6].

Proof. Obviously, it suffices to prove the result for $n = 2$. Assume that R is a ring containing I as an ideal, and that $R = \mathrm{End}(M_S)$ for some module M_S. (Of course we could take $S = M = R$.) Then $M_2(R) \cong \mathrm{End}(M_S \oplus M_S)$. We will verify condition (b) of Theorem 1.2. So assume that

$$X = M_1 \oplus M_2 \oplus Y = N_1 \oplus N_2$$

is an $M_2(I)$-admissible decomposition of X, where $M_1 \cong M_2 \cong M$. Let $\pi = \pi_1 + \pi_2 \in E := \mathrm{End}(X_S)$ be the projection onto $M_1 \oplus M_2$, and τ_i the projections onto N_i, $i = 1, 2$. Identifying $M_2(R)$ with $\pi E \pi$, this means that $\pi_i \tau_2 \pi_j \in e_{ij} I$, where e_{ij} are the canonical matrix units of $M_2(R)$. In particular, $\pi_1 \tau_2 \pi_1 \in e_{11} I$, and $e_{11} = \pi_1 = \pi_1 \tau_1 \pi_1 + \pi_1 \tau_2 \pi_1$, so that the exchange property of I gives us orthogonal idempotents $\upsilon_1, \upsilon_2 \in \pi_1 E \pi_1$ with $\upsilon_1 + \upsilon_2 = \pi_1$ such that $\upsilon_i = \alpha_i \tau_i \pi_1$ for some $\alpha_i \in \pi_1 E \pi_1$. Clearly we can assume that $\alpha_i = \upsilon_i \alpha_i$. Note that υ_2 and α_2 are in $e_{11} I$. Now take $\mu_i = \tau_i \alpha_i \tau_i \in E$. Then μ_i are orthogonal idempotents in E, and $\mu_i \in \tau_i E \tau_i$. Set $\mu_i' = \tau_i - \mu_i$, and write $N_i' = \mu_i' X \subseteq N_i$, for $i = 1, 2$.

We claim that $X = M_1 \oplus N_1' \oplus N_2'$. First note that, for every $x \in X$, we have

$$x = (\alpha_1 x_1 + \alpha_2 x_2) + (x_1 - \alpha_1 x_1) + (x_2 - \alpha_2 x_2) + \omega, \tag{1}$$

where $x_i = \mu_i x$ and $\omega = \mu_1' x + \mu_2' x$. Clearly, $\alpha_1 x_1 + \alpha_2 x_2 \in M_1$, and $\omega \in N_1' \oplus N_2'$. Since $\tau_i \alpha_i \mu_i = \tau_i \alpha_i \tau_i \alpha_i \tau_i = \tau_i \mu_i$, we have $x_i - \alpha_i x_i \in \ker \tau_i$, and so $x_i - \alpha_i x_i \in N_j$, where $j = i + 1 \pmod 2$. Moreover $\mu_j \pi_1 = \tau_j \alpha_j \tau_j \pi_1 = \tau_j \upsilon_j$, so that $\mu_j \alpha_i = 0$, and $\mu_j(x_i - \alpha_i x_i) = \mu_j \mu_i x - \mu_j \alpha_i x_i = 0$. We conclude that $x_i - \alpha_i x_i \in (\tau_j - \mu_j)X = N_j'$ (where $j = i + 1$

(mod 2)), and so (1) shows that $X = M_1 + (N_1' \oplus N_2')$. Now take $x \in M_1 \cap (N_1' \oplus N_2')$. Then $x = \pi_1 x$ and $\mu_i x = 0$ for $i = 1, 2$. Therefore, $x = \pi_1 x = \sum v_i x = \sum v_i^2 x = \sum \alpha_i \tau_i \alpha_i \tau_i \pi_1 x = \sum \alpha_i \mu_i \pi_1 x = 0$. Hence, we obtain $M_1 \cap (N_1' \oplus N_2') = 0$, and therefore $X = M_1 \oplus (N_1' \oplus N_2')$, as claimed.

Now consider

$$X = M_1 \oplus M_2 \oplus Y = (M_1 \oplus N_1') \oplus N_2'. \qquad (**)$$

We want to prove that $(**)$ is an I-admissible decomposition with respect to the factor M_2, i.e., $\pi_2 \tau_2' \pi_2 \in e_{22} I$, where τ_2' is the projection onto N_2' with kernel $M_1 \oplus N_1'$. From (1) we get

$$\tau_2' x = \mu_2'(\mu_1 x - \alpha_1 \mu_1 x) + \mu_2'(\mu_2 x - \alpha_2 \mu_2 x) + \mu_2' x. \qquad (2)$$

By using (2), one can check that $\tau_2' = \tau_2 - \tau_2 \alpha_1 \tau_1 - \tau_2 \alpha_2 \tau_2$, and therefore

$$\pi_2 \tau_2' \pi_2 = \pi_2 \tau_2 \pi_2 - \pi_2 \tau_2 \alpha_1 \tau_1 \pi_2 - \pi_2 \tau_2 \alpha_2 \tau_2 \pi_2. \qquad (3)$$

Now $\pi_2 \tau_2 \pi_2 \in e_{22} I$ by hypothesis, and $\pi_2 \tau_2 \alpha_i \tau_i \pi_2 = (\pi_2 \tau_2 \pi_1)(\pi_1 \alpha_i \tau_i \pi_2) \in (e_{21} I)(e_{12} R) = e_{22} I$, so that we deduce from (3) that $\pi_2 \tau_2' \pi_2 \in e_{22} I$. Let $\pi' \in E$ be the projection onto $M_2 \oplus Y$ with kernel M_1. Then $\pi' X = M_2 \oplus Y = \pi' N_1' \oplus \pi' N_2'$. We will identify $\operatorname{End}(\pi' X)$ with $\pi' E \pi'$. Let τ_1'' and τ_2'' be the projections onto $\pi' N_1'$ and $\pi' N_2'$, respectively. Note that $\tau_2'' = \pi' \tau_2' \pi'$. Since $\pi_2 = \pi' \pi_2 = \pi_2 \pi'$, we have $\pi_2 \tau_2'' \pi_2 = \pi_2 (\pi' \tau_2' \pi') \pi_2 = \pi_2 \tau_2' \pi_2 \in e_{22} I$. By hypothesis, there exist $N_1'' \subseteq \pi' N_1'$ and $N_2'' \subseteq \pi' N_2'$ such that $\pi' X = M_2 \oplus N_1'' \oplus N_2''$. Now π' induces isomorphisms from N_1' onto $\pi' N_1'$ and from N_2' onto $\pi' N_2'$. Let S_1 and S_2 be the inverse images of N_1'' and N_2'', respectively. Then $X = M_1 \oplus M_2 \oplus S_1 \oplus S_2$ and $S_i \subseteq N_i$, as desired. ∎

We close this section by noting two important consequences of the exchange property for rings without unit. Again, these properties were already known in the unital case (see [2]). Let I be a ring and let R be a unital ring containing I as an ideal. Let $FP(I, R)$ denote the class of all finitely generated projective R-modules P such that $P = PI$. Let $V(I)$ be the set of all the isomorphism classes of elements from $FP(I, R)$. $V(I)$ is an abelian monoid under addition given by $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$. As observed in [13, p. 296], $V(I)$ depends only on the structure of I as a ring without unit. (In fact an alternative description of $V(I)$ is obtained by considering the set of all equivalence classes of idempotents in $M_\infty(I)$, cf. Section 3.)

A monoid M is said to be a *refinement monoid* (e.g., [20]) if whenever $a + b = c + d$ in M, there exist $x, y, z, t \in M$ such that $a = x + y$ and $b = z + t$ while $c = x + z$ and $d = y + t$. (When applied to $V(I)$, this means that we can "refine" two different direct sum decompositions of a given $P \in FP(I, R)$.)

PROPOSITION 1.5. *Let I be an exchange ideal of a unital ring R. Then the following properties hold*:

(a) *Every $P \in FP(I, R)$ is a direct sum of cyclic submodules.*

(b) *$V(I)$ is a refinement monoid.*

Proof.

(a) Let $P \in FP(I, R)$. There exist a positive integer n and submodules Q, P' of nR such that $nR = Q \oplus P'$ and $P \cong P'$. The decomposition

$$nR = R \oplus (n-1)R = Q \oplus P'$$

is clearly I-admissible, and so there exist submodules $Q_1 \subseteq Q$ and $P_1 \subseteq P'$ such that $nR = R \oplus Q_1 \oplus P_1$. Write $Q = Q_1 \oplus Q_2$ and $P' = P_1 \oplus P_2$, for some P_2, Q_2. Then $Q_2 \oplus P_2 \cong R$ and so P_2 is a cyclic module. On the other hand we have $(n-1)R \cong Q_1 \oplus P_1$. So induction gives the result.

(b) Assume that $X = A \oplus B = C \oplus D$, with $A, B, C, D \in FP(I, R)$. We will see that $\mathrm{End}_R(A)$ is an exchange ring, and so A has the exchange property [19, Theorem 2]. There exists an idempotent e in $M_n(I)$, for some $n \geq 1$, such that $A \cong e(R^n)$. Consequently $\mathrm{End}_R(A) \cong eM_n(R)e = eM_n(I)e$, which is a unital exchange ring by Theorem 1.4 and Proposition 1.3.

Since A has the exchange property, the proof in [2, Proposition 1.2] applies. ∎

2. EXTENSIONS OF EXCHANGE RINGS

We start with a useful lemma. Very roughly, the lemma says that regular elements can be lifted modulo an exchange ideal provided idempotents can be lifted.

LEMMA 2.1. *Let I be an exchange ideal of a unital ring R. Write $\overline{R} = R/I$, and denote by $\bar{t}$ the image of $t \in R$ under the canonical map $R \to \overline{R}$. Let p be an idempotent in R, and let x, y be elements in R such that $\bar{p} = \overline{xy}$. Then there exist $a \in pRx$ and $b \in yRp$ such that $a = aba$ and $b = bab$, while $\bar{a} = \overline{px}$ and $\bar{b} = \overline{yp}$.*

Proof. Note that $\bar{p} = \overline{pxyp}$, and so $p - pxyp \in pRp \cap I = pIp$. By Proposition 1.3, pIp is an exchange ideal of pRp, and so there exists an idempotent $q \in pIp$ such that $q \in (p - pxyp)R$ and $p - q \in pxyp(pRp)$. Let r be an element in $pR(p-q)$ such that $p - q = pxyr$. Since $q \in I$ we get

$$\bar{p} = \bar{p} - \bar{q} = \overline{pxyr} = \overline{pr} = \bar{r}.$$

Define $a := (p-q)x \in pRx$ and $b := yr \in yRp$. Then $a = aba$ and $b = bab$ follow from the relation $\overline{p-q} = \overline{pxyr} = \overline{(p-q)xyr(p-q)}$. Moreover $\bar{a} = (\bar{p}-\bar{q})\bar{x} = \overline{px}$ and $\bar{b} = \overline{yr} = \overline{yp}$, as desired. ∎

We are now ready to obtain our main result, which describes the exact conditions needed for an extension of two rings to be an exchange ring. Given two idempotents e and f of a ring, we will write $e \leq f$ in case $e = ef = fe$.

THEOREM 2.2. *Let I be an ideal of the (possibly non-unital) ring L. Then L is an exchange ring if and only if I and L/I are exchange rings and idempotents can be lifted modulo I.*

Proof. Let R be a unital ring containing both I and L as ideals. (For example, we could take $R = L^1$.)

Assume that L is an exchange ring. By applying Lemma 1.1 with L playing the role of I, we see that L/I is an exchange ring, and that idempotents can be lifted from L/I to L. The fact that I is an exchange ring is straightforward (cf. Example 1 after Theorem 1.2).

Conversely, assume that I and L/I are exchange rings and idempotents lift from L/I to L. Take an element x in L. Write $\overline{R} = R/I$ and denote by $\bar{t}$ the image of an element $t \in R$ under the canonical map $\pi\colon R \to R/I$. Since $\overline{L} := L/I$ is an exchange ideal of R/I, there exists an idempotent $e \in \overline{L}$ such that $e \in \bar{x}\overline{L}$ and $1 - e \in (1-\bar{x})\overline{R}$. Take $y \in \overline{L}e$ such that $e = \overline{xy}$, and note that $\overline{yx}$ is an idempotent in $\overline{L}$. Since idempotents lift from L/I to L, we can find an idempotent $p \in L$ such that $\bar{p} = \overline{yx}$. Now, apply Lemma 2.1 to get $a \in pRy$ and $b \in xRp$ such that $a = aba$ and $b = bab$, while $\bar{a} = \overline{py}$ and $\bar{b} = \overline{xp}$. Set $q = ba$ and note that q is an idempotent in L such that $\bar{q} = \bar{b}\bar{a} = \overline{xpy} = e$. Moreover, since $b \in xL$, we see that $q = ba \in xL$.

Since $\bar{q} = e$, we have $1 - \bar{q} = 1 - e \in (1-\bar{x})\overline{R}$. Write $1 - \bar{q} = (1 - \bar{x})\bar{z}$, with $z = z(1-q)$. Note that $\bar{z} = 1 - \bar{q} + \overline{xz} \in 1 + \overline{L}$, and so $\bar{z}(1-\bar{x})$ is an idempotent in $\overline{R}$ of the form $1 - t$, where t is an idempotent in $\overline{L}$. Consequently, $\bar{z}(1-\bar{x})$ can be lifted to an idempotent $p_1 \in R$. Now we apply Lemma 2.1 to the equation $\overline{p_1} = \bar{z}(1-\bar{x})$ to obtain $a_1 \in p_1Rz$ and $b_1 \in (1-x)Rp_1$ such that $a_1 = a_1b_1a_1$ and $b_1 = b_1a_1b_1$, while $\overline{a_1} = \overline{p_1z}$ and $\overline{b_1} = \overline{(1-x)p_1}$. Set $q_1 = (1-q)b_1a_1$ and note that $\overline{q_1} = 1 - \bar{q}$. Observe that $a_1(1-q) = a_1$, because $a_1 \in p_1Rz$ and $z = z(1-q)$. Therefore, $q_1 = (1-q)b_1a_1 = (1-q)b_1a_1(1-q)$ is an idempotent, and $q_1 \leq 1 - q$. This implies that $h := 1 - (q + q_1)$ is an idempotent in I.

By Proposition 1.3, $hRh = hIh$ is a unital exchange ring, so that there exists an idempotent $r' \in hRh$ such that

$$r' \in (hxh)(hRh) \tag{1}$$

and

$$h - r' \in (h - hxh)(hRh). \tag{2}$$

Write $r' = (hxh)c$, where $c \in hRr'$. Then $r := (xhc)h = xhc$ is an idempotent in Rh with $r \in xR$. Note also that $r' = hr$.

Finally, set $k = q + (1 - q)r$, and note that $k = k^2$, and $k \in xR$ because $q, r \in xR$. By Lemma 1.1(3), to end the proof we need only to check that $kR + (1 - x)R = R$. Write $A = kR + (1 - x)R$. To see that $A = R$, it suffices to prove that $h, 1 - h \in A$. Since $q + (1 - q)r = k \in A$ and $r = r(1 - q)$, we obtain that $q, r \in A$. Since $q_1 \in (1 - q)(1 - x)R$ and $q, 1 - x \in A$, we get that $q_1 \in A$, so that $1 - h = q + q_1 \in A$. Now it follows from the relation $hr = r - (1 - h)r \in A$ that $r' = hr \in A$. Note also that

$$h(1 - x) = (1 - x) - (1 - h)(1 - x) \in A,$$

so that, by using (2), we have $h - r' \in (h - hxh)R \subseteq h(1 - x)R \subseteq A$. We get $h = r' + (h - r') \in A$, and so $h, 1 - h \in A$ and $A = R$, as desired. ∎

The result of Nicholson [14, Proposition 1.5] can now be obtained as a corollary of our theorem.

COROLLARY 2.3. *Let R be a unital ring with Jacobson radical J. Then R is an exchange ring if and only if R/J is an exchange ring and idempotents can be lifted modulo J.*

Proof. Since J is an exchange ring, the result follows from Theorem 2.2. ∎

Next, we note two special cases in which lifting of idempotents is automatic.

COROLLARY 2.4. *Let I be a π-regular ring and let L be any ring containing I as an ideal. Then L is an exchange ring if and only if L/I is an exchange ring.*

Proof. By [12, Lemma 3], the idempotents lift modulo π-regular ideals. Hence, the result follows from Theorem 2.2. ∎

COROLLARY 2.5. *Let L be a ring with an exchange ideal I, and assume that L/I is a radical ring. Then L is an exchange ring.*

Proof. Since L/I is a radical ring, it is an exchange ring without idempotents. So the result follows from Theorem 2.2. ∎

3. K-THEORETIC CHARACTERIZATION

In this section we will prove that the lifting of idempotents modulo an exchange ideal can be checked from K-theoretic data. As a consequence, we obtain an improvement of Theorem 2.2.

Two idempotents e and f of a ring I are said to be *equivalent*, written $e \sim f$, if there exist $x \in eIf$ and $y \in fIe$ such that $e = xy$ and $f = yx$. Note that, for $e, f \in I$, we have $e \sim f$ if and only if $eL \cong fL$ for every (some) ring L containing I as an ideal.

LEMMA 3.1. *Let I be an exchange ideal of a unital ring R. Denote by $\bar{r}$ the image of $r \in R$ under the canonical map $R \to R/I$.*

(a) *If p is an idempotent of R such that $\bar{p} \sim e$ for some idempotent $e \in \overline{R}$, then there exists an idempotent $q \in R$ such that $q \sim p - p_0$ for some idempotent $p_0 \in pIp$, and $\bar{q} = e$.*

(b) *If p, q are idempotents in R such that $\bar{q} \le \bar{p}$, then there exists an idempotent q' in R such that $q' \le p$ and $\bar{q}' = \bar{q}$.*

(c) *If $p, q \in R$ are idempotents such that $\bar{p} \sim \bar{q}$, then there exist idempotents $p_1 \le p$ and $q_1 \le q$ such that $p_1 \sim q_1$ and $\overline{p_1} = \bar{p}, \overline{q_1} = \bar{q}$.*

Proof.

(a) Write $e = \overline{yx}$ and $\bar{p} = \overline{xy}$ for some $x \in pR$ and $y \in Rp$. By Lemma 2.1, there exist $a \in pRx$ and $b \in yRp$ such that $a = aba$ and $b = bab$, while $\bar{a} = \overline{px} = \bar{x}$ and $\bar{b} = \overline{yp} = \bar{y}$. Take $q = ba$. Then q is an idempotent in R such that $\bar{q} = e$ and $q = ba \sim ab \le p$ with $p - ab \in I$.

(b) Set $x = qp$ and $y = pq$. Since $\bar{q} \le \bar{p}$, we have

$$\bar{q} = \bar{x} = \bar{y} = \overline{xy},$$

and so Lemma 2.1 gives us elements $a \in qRx$ and $b \in yRq$ such that $a = aba$ and $b = bab$, while $\bar{a} = \overline{qx} = \bar{q}$ and $\bar{b} = \overline{yq} = \bar{q}$. Set $q' = ba$. Then $q' \le p$ and $\bar{q}' = \bar{q}$.

(c) Write $\bar{p} = \overline{xy}$ and $\bar{q} = \overline{yx}$ for some $x \in pRq$ and $y \in qRp$. By Lemma 2.1, there exist $a \in pRx$ and $b \in yRq$ such that $a = aba$ and $b = bab$, while $\bar{a} = \bar{x}$ and $\bar{b} = \bar{y}$. The result follows by taking $p_1 = ab$ and $q_1 = ba$. ∎

Recall that the Grothendieck group $K_0(R)$ of a unital ring R is the Grothendieck group of the abelian monoid $V(R)$ of isomorphism classes of finitely generated projective right R-modules [16, p. 5]. It will be convenient for us to use the description of $V(R)$ from idempotents, as given in [16], for example. Therefore we will identify $V(R)$ with the set of equivalence classes of idempotents in $M_\infty(R) := \varinjlim M_n(R)$, where the maps $M_n(R) \to M_m(R)$, for $n \le m$, are defined by sending $x \in M_n(R)$ to $\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_m(R)$. For $e \in M_\infty(R)$, we will denote the class of e in $K_0(R)$ by $[e]$. For $e, f \in M_\infty(R)$, set $e \oplus f = \left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right) \in M_\infty(R)$, and $1_r = 1 \oplus \cdots \oplus 1$ (r times).

For any unital ring R, the group $K_0(R)$ has a natural pre-order, obtained by taking as the positive cone the equivalence classes in $K_0(R)$ of the finitely generated projective modules. In terms of the idempotent picture, we set

$$K_0(R)^+ = \left\{[e] \middle| e = e^2 \in M_\infty(R)\right\}.$$

We will fix the following notation for our next results. Let I be an exchange ideal of a unital ring R. We will denote by $\pi\colon M_\infty(R) \to M_\infty(R/I)$ the canonical projection, while $\Psi\colon K_0(R) \to K_0(R/I)$ will stand for the induced map on Grothendieck groups.

LEMMA 3.2. *Let I be an exchange ideal of a unital ring R. If e is an idempotent in $M_n(R/I)$ for some $n \ge 1$, such that $[e] \in \Psi(K_0(R))$, then there exists an idempotent p in $M_n(R)$ such that $\pi(p) = e$. In particular,*

$$\Psi\left(K_0(R)^+\right) = \Psi(K_0(R)) \cap K_0(R/I)^+.$$

Proof. Let e be an idempotent in $M_n(R)$ for some $n \ge 1$, and assume that $[e] \in \Psi(K_0(R))$. There exists an element $T \in K_0(R)$ such that $\Psi(T) = [e]$. Write $T = [g] - [h]$ for some idempotents g, h in $M_\infty(R)$. Now $\Psi([g] - [h]) = [e]$ gives us the relation

$$\pi(g) \oplus 1_r \sim e \oplus \pi(h) \oplus 1_r$$

for some non-negative integer r. Changing g to $g \oplus 1_r$ and h to $h \oplus 1_r$, we can assume that

$$\pi(g) \sim e \oplus \pi(h).$$

Now passing to $M_m(R)$ for $m \ge n$ big enough, we can assume g, h are idempotents in $M_m(R)$ and that e and $\pi(h)$ are orthogonal idempotents in $M_m(R/I)$. Set $S = M_m(R)$ and $I' = M_m(I)$. By Theorem 1.4, I' is an exchange ideal of S. Now g is an idempotent in S and $\pi(g) \sim e + \pi(h)$, so that Lemma 3.1(a) gives us an idempotent q in S such that $\pi(q) = e + \pi(h)$. Since $\pi(h) \le \pi(q)$, we obtain using Lemma 3.1(b) an idempotent $h' \in S$ such that $h' \le q$ and $\pi(h') = \pi(h)$. Put $q' := q - h'$. Then q' is an idempotent in S and $\pi(q') = \pi(q) - \pi(h') = e + \pi(h) - \pi(h) = e$. Since $\pi(q') = e \le \pi(1_n)$ a further use of Lemma 3.1(b) gives us an idempotent $p \le 1_n$ such that $\pi(p) = \pi(q') = e$. Thus $p \in M_n(R)$ is an idempotent such that $\pi(p) = e$, as desired. ∎

THEOREM 3.3. *Let I be an exchange ideal of a unital ring R. Then Ψ: $K_0(R) \to K_0(R/I)$ is surjective if and only if the idempotents of $M_n(R/I)$ can be lifted to idempotents of $M_n(R)$ for all $n \geq 1$. If every finitely generated projective R/I-module is isomorphic to a direct sum of cyclic modules, then Ψ is surjective if and only if the idempotents of R/I can be lifted to idempotents of R.*

Proof. The result follows from Lemma 3.2. ∎

Let I be a ring without unit. The group $K_0(I)$ is then defined as the kernel of the natural map $K_0(I^1) \to K_0(\mathbb{Z})$, where $I^1 = I \oplus \mathbb{Z}$ is the unitization of I [16, 1.5].

In order to obtain a K-theoretic version of Theorem 2.2, we need the following easy lemma.

LEMMA 3.4. *Let I be an ideal of the exchange ring L. Then the idempotents of $M_n(L^1/I)$ can be lifted to idempotents of $M_n(L^1)$, for all $n \geq 1$.*

Proof. Note that $L^1/I \cong (L/I)^1$. Write $A = L/I$, and note that A is an exchange ring. We will prove that every idempotent in $M_\infty(A^1)$ is equivalent to an idempotent of the form $(1_r - g) \oplus h$ for some $r \geq 0$, where g and h are idempotents in $M_\infty(A)$ and $g \leq 1_r$.

Let ρ: $M_\infty(A^1) \to M_\infty(\mathbb{Z})$ be the canonical projection, and let $e \in M_\infty(A^1)$ be an idempotent. There exists $r \geq 0$ such that $\rho(e) \sim 1_r = \rho(1_r)$. By Lemma 3.1(c), there exist orthogonal decompositions $e = e' + h$ and $1_r = e'' + g$ such that $e' \sim e''$ and g, h are idempotents in $M_\infty(A)$. Now we obtain

$$e \sim e' \oplus h \sim e'' \oplus h = (1_r - g) \oplus h.$$

Since $M_n(I)$ is an exchange ideal of $M_n(L)$ for all $n \geq 1$, it is clear that all idempotents of the form $(1_r - g) \oplus h$, with $g \leq 1_r$ and g, h idempotents in $M_\infty(L/I)$ can be lifted to idempotents in $M_\infty(L^1)$. By Lemma 3.2, we get that, for all $n \geq 1$, all idempotents of $M_n(L^1/I)$ can be lifted to idempotents of $M_n(L^1)$, as desired. ∎

Now, we are ready to give an improvement of Theorem 2.2.

THEOREM 3.5. *Let I be an ideal of the (possibly non-unital) ring L. Then L is an exchange ring if and only if I and L/I are exchange rings and the canonical homomorphism $K_0(L) \to K_0(L/I)$ is surjective.*

Proof. If L is an exchange ring then I and L/I are exchange rings. Also, by Lemma 3.4, the idempotents of $M_\infty(L^1/I)$ can be lifted to idempotents of $M_\infty(L^1)$, showing in particular that the map $K_0(L^1) \to K_0(L^1/I)$ is surjective. Now, the Five Lemma gives that the map $K_0(L) \to K_0(L/I)$ is also surjective.

Now assume that I and L/I are exchange rings and that the map $K_0(L) \to K_0(L/I)$ is surjective. Since the homomorphism $K_0(L) \to K_0(L/I)$ is surjective, the map $K_0(L^1) \to K_0(L^1/I)$ is also surjective, again by the Five Lemma. So we conclude from Theorem 3.3 that idempotents of L^1/I lift to idempotents of L^1. In particular, idempotents of L/I lift to idempotents of L, and so it follows from Theorem 2.2 that L is an exchange ring. ∎

Following [8, p. 116], we say that a regular right self-injective ring R is *purely infinite* if it contains no nonzero directly finite central idempotents. We say that a right self-injective ring R is *purely infinite* if $R/J(R)$ is purely infinite, where $J(R)$ is the Jacobson radical of R. (Recall that $R/J(R)$ is a regular right self-injective ring [17, Chap. XIV, Corollary 1.3(ii)].)

COROLLARY 3.6. *Let I be an exchange ideal of a unital ring R. If R/I is a purely infinite right self-injective ring, then R is an exchange ring.*

Proof. Since $\overline{R} := R/I$ is right self-injective, idempotents of $\overline{R}/J(\overline{R})$ lift to $\overline{R}$ [17, Chap. XIV, Corollary 1.5], and $K_0(\overline{R}) = K_0(\overline{R}/J(\overline{R}))$. Moreover, $K_0(\overline{R}/J(\overline{R})) = 0$ by [8, Proposition 15.6]. So $K_0(\overline{R}) = 0$ and therefore the map $\Psi\colon K_0(R) \to K_0(R/I)$ is obviously surjective. It follows from Theorem 3.5 that R is an exchange ring. ∎

Recall that a ring R is called *semilocal* if $R/J(R)$ is semisimple artinian, while R is said to be *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. (See, for example, [11, Chap. 7].) Note that the semiperfect rings are exactly the semilocal exchange rings.

COROLLARY 3.7. *Let R be a semilocal ring. Then R is semiperfect if and only if the map $K_0(R) \to K_0(R/J(R))$ is an isomorphism.*

Proof. The map $K_0(R) \to K_0(R/J(R))$ is always injective. So the result follows from Theorem 3.5. ∎

Finally we extend the characterization given in [2, Theorem 7.2] to the non-unital case, and we show how the known extension theorems for C^*-algebras of real rank zero follow from our results. We refer the reader to [5] for the definition and basic properties of C^*-algebras of real rank zero. As in [5], we will denote the real rank of a C^*-algebra A by $RR(A)$. Recall that, by definition, a non-unital C^*-algebra A has real rank zero if and only if its C^*-algebra unitization $\tilde{A} = A \oplus \mathbb{C}$ has real rank zero. Consequently, we obtain immediately from [2, Theorem 7.2] and Theorem 2.2 the following:

THEOREM 3.8. *The C^*-algebras with real rank zero are precisely those C^*-algebras which are exchange rings.*

As a consequence of Theorem 3.8, we can see some known results on C^*-algebras of real rank zero as corollaries of our main results.

COROLLARY 3.9 [5, 3.14]. *If I is a closed ideal in a C^*-algebra A, then $RR(A) = 0$ if and only if $RR(I) = RR(A/I) = 0$ and every projection in A/I lifts to a projection in A.*

Proof. By Theorems 3.8 and 2.2, $RR(A) = 0$ if and only if $RR(I) = RR(A/I) = 0$ and idempotents can be lifted modulo I. Now, the result follows from the well-known fact that idempotents lift modulo I if and only if projections in A/I lift to projections in A (see [4, 4.6.2]). ∎

COROLLARY 3.10 [5, 3.15]. *If I is a closed ideal in a C^*-algebra A, then $RR(A) = 0$ if and only if $RR(I) = RR(A/I) = 0$ and the canonical map $K_0(A) \to K_0(A/I)$ is surjective.*

Proof. The result follows immediately from Theorems 3.8 and 3.5. ∎

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